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GELFOND'S METHOD FOR ALGEBRAIC INDEPENDENCE

BY

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ABSTRACT. This paper extends Gelfond's method for algebraic independence to fields K with transcendence type $\leq \tau$. The main results show that the elements of a transcendence basis for K and at least two more numbers from a prescribed set are algebraically independent over Q. The theorems have a common hypothesis: $\{\alpha_1, \ldots, \alpha_M\}$, $\{\beta_1, \ldots, \beta_N\}$ are sets of complex numbers, each of which is Q-linearly independent.

THEOREM A. *If* $(2\tau - 1) < MN$, *then at least two of the numbers* $\alpha_i, \beta_j, \exp(\alpha_i\beta_j)$, $1 \leq i \leq M$, $1 \leq j \leq N$, *are algebraically dependent over* K.

THEOREM B. *If* $2\tau(M + N) \leq MN + M$, *then at least two of the numbers* $\alpha_i, \exp(\alpha_i\beta_j)$, $1 \leq i \leq M$, $1 \leq j \leq N$, *are algebraically dependent over* K.

THEOREM C. *If* $2\tau(M + N) \leq MN$, *then at least two of the numbers* $\exp(\alpha_i\beta_j)$, $1 \leq i \leq M$, $1 \leq j \leq N$, *are algebraically dependent over* K.

I. Introduction and statement of results. The results of this paper are based upon the work of A. O. Gelfond and S. Lang. In 1949 Gelfond [10], [11] published two theorems showing that certain sets of numbers related by the exponential function cannot lie in a field of transcendence degree one over the rational numbers. Since then A. A. Shmelev [17], [18] found another case to which Gelfond's method applied and R. Tijdeman [23] and M. Waldschmidt [26], independently, removed an extraneous hypothesis in Gelfond's method. Another type of theorem (Theorem 3 below) to which Gelfond's method applies was found, independently, by Shmelev [19] Waldschmidt [26], R. Wallisser [6], and the author [3]. So the published results using Gelfond's method in $\mathbf{C}$ fall into the following form, with minimal choices for M and N noted in parentheses for convenience.

COMMON HYPOTHESIS. *Let the sets of complex numbers* $\{\alpha_1, \cdots, \alpha_M\}$ *and* $\{\beta_1, \cdots, \beta_N\}$ *each be linearly independent over* $\mathbf{Q}$.

THEOREM 1. *If* $M + N < MN$, *then at least two of the numbers*

$$\alpha_i, \quad \beta_j, \quad \exp(\alpha_i\beta_j), \quad i = 1, \cdots, M; \ j = 1, \cdots, N,$$

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are algebraically independent over $\mathbf{Q}$ $(M = 2,\ N = 3;\ M = 3, N = 2)$.

THEOREM 2. *If* $2(M + N) \leqslant MN + M$, *then at least two of the numbers*

$$\alpha_i,\quad \exp(\alpha_i\beta_j),\quad i = 1, \cdots, M;\ j = 1, \cdots, N,$$

are algebraically independent over $\mathbf{Q}$ $(M = N = 3;\ M = 4,\ N = 2)$.

THEOREM 3. *If* $2(M + N) \leqslant MN$, *then at least two of the numbers*

$$\exp(\alpha_i\beta_j),\quad i = 1, \cdots, M;\ j = 1, \cdots, N,$$

are algebraically independent over $\mathbf{Q}$ $(M = 3,\ N = 6;\ M = N = 4;\ M = 6, N = 3)$.

Note that, say, α_1 (or β_1) may be assumed to be 1 without loss of generality by dividing the α's and multiplying the β's by the original α_1 (or $1/\beta_1$).

In 1966 Lang [13] proved that certain values of the exponential function cannot all be algebraic over a field with transcendence type $\leqslant \tau$. Since we shall recall the exact definition of transcendence type below in §III, let us be content here to say only that τ is an asymptotic measure of how small polynomials in a transcendence basis with integer coefficients might be in terms of the degree and maximum absolute value of the coefficients. More specifically Lang proved the following analogue of Theorem 3:

THEOREM 4. *Under the Common Hypothesis, if* K *is a field of transcendence type* $\leqslant \tau$, $2 \leqslant \tau$, *and* $\tau(M + N) \leqslant MN$, *then at least one of the numbers*

$$\exp(\alpha_i\beta_j),\quad i = 1, \cdots, M;\ j = 1, \cdots, N,$$

is not algebraic over K.

In the author's thesis (Cornell, 1970) the following theorems, announced in [2], analogous to Theorem 1 and Theorem 2 were established:

THEOREM 5. *Under the Common Hypothesis, if* $\tau(M + N) < MN + M + N$ *and* K *is a field with transcendence type* $\leqslant \tau$, *then at least one of the numbers*

$$\alpha_i, \beta_j,\quad \exp(\alpha_i\beta_j),\quad i = 1, \cdots, M;\ j = 1, \cdots, N,$$

is not algebraic over K.

THEOREM 6. *Under the Common Hypothesis, if* $\tau(M + N) \leqslant MN + M$ *and* K *is a field with transcendence type* $\leqslant \tau$, *then at least one of the numbers*

$$\alpha_i, \quad \exp(\alpha_i\beta_j), \quad i = 1, \cdots, M; \ j = 1, \cdots, N,$$

is not algebraic over K.

Theorems 5 and 6 generalize the classical Hermite-Lindemann and Gelfond-Schneider theorems, since $\mathbf{Q}$ is a field with transcendence type $\leqslant 1$. Moreover their proofs are modeled on the classical proofs. These theorems were rediscovered by Waldschmidt [26] who axiomatized the situation even more. Since their proofs are now in the literature [27], it is not necessary to repeat them here. Depending on the taste or unfamiliarity of the reader, he might find it desirable after reading Lang's proof of Theorem 4 to attempt his own proofs of Theorems 5 and 6 patterned after the classical models in say [13] or [16].

The main results of the present paper, modulo a beautiful result of Tijdeman (Theorem 5.3 below), are also from the author's thesis. This investigation was triggered by an intriguing remark of Lang [13, pp. 55–56] to the effect that although Lang had obtained Theorem 4 by generalizing from the case $\mathbf{Q} = K$ and $\tau = 1$, Gelfond could not state a similar generalization for his method. Tijdeman's result mentioned above takes care of one of the main difficulties which Lang points out in generalizing Gelfond's method. The other main difficulty is a question about sequences of "diophantine approximations" over a field with transcendence type $\leqslant \tau$. That question has been built into the hypotheses of one of Waldschmidt's main theorems of [27]. The goal of the present paper is to settle the question about sequences and establish Gelfond's method over fields with transcendence type $\leqslant \tau$. In previous papers the author [4] and Waldschmidt [26], independently, sharpened Gelfond's characterization of transcendental numbers [11, p. 148] in order to solve Schneider's "Eighth Problem" [16, p. 138], [5], [24], [28]. In §IV, we generalize this sharpened characterization to characterize numbers which are transcendental over some fixed field with transcendence type $\leqslant \tau$.

For those readers acquainted with Waldschmidt's general axiomatization of transcendence proofs [27], it is necessary only to read the first four sections to see, in Waldschmidt's terminology, that if $K \subseteq \mathbf{C}$ is a field of transcendence type $\leqslant \tau$ over $\mathbf{Q}$ and if L is an extension field of K of transcendence degree at most one over K, then L is a $\mathbf{Q}^{(2\tau-1)}$ field. However, for the sake of the nonexpert and to avoid relying on the extreme compactness of Waldschmidt's theorems [27, §6], we shall follow the tradition in the theory of transcendental numbers of giving self-contained proofs whenever possible.

The theorems that we shall prove are as follows:

New Common Hypothesis. *Let* K *be a field with transcendence type* $\leqslant \tau$ *and let* $\{\alpha_1, \cdots, \alpha_M\}$ *and* $\{\beta_1, \cdots, \beta_N\}$ *be sets of complex numbers,* *each of which is linearly independent over* $\mathbf{Q}$.

THEOREM A. *If* $2\tau(M+N) < MN + M + N$, *then at least two of the numbers* $\alpha_i, \beta_j, \exp(\alpha_i\beta_j)$, $i = 1, \cdots, M$; $j = 1, \cdots, N$, *are algebraically independent over* K.

THEOREM B. *If* $2\tau(M+N) \leqslant MN + M$, *then at least two of the numbers* $\alpha_i, \exp(\alpha_i\beta_j)$, $i = 1, \cdots, M$; $j = 1, \cdots, N$, *are algebraically independent over* K.

THEOREM C. *If* $2\tau(M+N) \leqslant MN$, *then at least two of the numbers* $\exp(\alpha_i\beta_j)$, $i = 1, \cdots, M; j = 1, \cdots, N$, *are algebraically independent over* K.

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After this manuscript was written, the author became aware of work of Shmelev [20] which essentially considers four interesting special cases of Theorem B. The approach, although also based on Gelfond's method, is substantially different from that offered here. Theorem B gives strengthenings of each of Shmelev's four theorems.

W. Adams [1] and T. Shorey [21] have developed Gelfond's method for algebraic independence in p-adic completions of $\mathbf{Q}$. The results of this paper have similar extensions to the p-adic domain.

Moreover, Shmelev [19] has investigated quantitative implications of Gelfond's method.

II. Size of elements in a finitely generated extension of the rationals. Ordinary height and degree [16], loosely speaking, measure the complexity of algebraic numbers. Analogously, "size" is an attempt to measure the complexity of numbers in nonalgebraic, but finitely generated, subfields of $\mathbf{C}$. This section parallels much of Chapter V, §2 of Lang's book *Introduction to transcendental numbers.* (See also [11], [23], [27].) The only essential deviation arises from trying to define "size" so that it is more obviously related to "absolute size" through (2.7), in such a way that the proofs remain uncluttered. Unfortunately size is still not intrinsic, but rather depends on the bases involved. However our definition does have the aesthetic advantage that no transcendental number with large "absolute size" can have small "size".

For the rest of this paper, let $K \subseteq \mathbf{C}$ be a *finitely generated extension field of the rational numbers such that* $K = \mathbf{Q}(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with

(2.1) (1) $\{x_1, \cdots, x_m\}$ *a transcendence basis for* K *over* $\mathbf{Q}$, *and*
(2) $\{y_1, \cdots, y_n\}$ *a (vector space) basis for* K *over* $\mathbf{Q}(x_1, \cdots, x_m)$.

Before we define "size" in general, let us first consider a special case:

Purely transcendental case. Let $K = \mathbf{Q}(x_1, \cdots, x_m)$ with $x_1, \cdots, x_m$ algebraically independent over $\mathbf{Q}$. Then think of

$$I_K = \mathbf{Z}[x_1, \cdots, x_m] \tag{2.2}$$

as the "ring of integers" of K. Designate an m-tuple of nonnegative integers $(i_1, \cdots, i_m)$ by ι. If $P \in I_K$ is nonzero, say

$$P = \sum A_\iota x^{i_1} \cdots x^{i_m}, \qquad A_\iota \in \mathbf{Z},$$

define

$$\begin{aligned} \deg P &= \deg_{x_1} P + \cdots + \deg_{x_m} P, \\ \operatorname{ht} P &= \max |A_\iota|, \\ \operatorname{size} P &= \max\{\deg P, \log \operatorname{ht} P\}, \end{aligned}$$

where by $\log$ we mean the natural logarithm. More generally if α in K is nonzero, then α is uniquely P/Q with P and Q in I_K with only ± 1 as common factors. Then define

$$\begin{aligned} \deg \alpha &= \max\{\deg P, \deg Q\}, \\ \operatorname{ht} \alpha &= \max\{\operatorname{ht} P, \operatorname{ht} Q\}, \\ \operatorname{size} \alpha &= \max\{\deg \alpha, \log \operatorname{ht} \alpha\}. \end{aligned} \tag{2.3}$$

The following lemma generalizes an earlier result of J. F. Koksma and J. Popken. K. Mahler [14], [15] has given an elegant alternate proof, and Tijdeman [23] has given a particularly smooth exposition of the first case needed for the inductive proof of equation (130′) of [11] which is the stronger result which Gelfond actually proved.

LEMMA 2.1 (GELFOND). *Let* $\alpha_1, \cdots, \alpha_k \in I_K$ *and* $\alpha = \alpha_1 \cdots \alpha_k$. *Then*

$$(\operatorname{ht} \alpha_1) \cdots (\operatorname{ht} \alpha_k) \leqslant (\operatorname{ht} \alpha) \exp(\deg \alpha). \tag{2.4}$$

This lemma implies that if we have *any* representation of α in K as $\alpha = R/S$ with $R, S \in I_K$, then size α cannot be much larger than the size of R and S. More precisely, since $\max\{\deg R, \deg S\} + \log \max\{\operatorname{ht} R, \operatorname{ht} S\} \leqslant 2 \max\{\operatorname{size} R, \operatorname{size} S\}$, it follows that

$$\operatorname{size} \alpha \leqslant 2 \max\{\operatorname{size} R, \operatorname{size} S\}.$$

General case. If $\alpha \in K$, then multiplication by α is a linear transformation on K. If α is nonzero let

$$(P_{ij}/Q) \tag{2.5}$$

be its matrix over $\mathbf{Q}(x_1, \cdots, x_m)$ with respect to the ordered basis $(y_1, \cdots, y_n)$ for K satisfying the condition that Q and the greatest common divisor of all the P_{ij} are relatively prime, i.e. have only ± 1 as common factors. Then call Q the *denominator* of α and then define

$$(2.6)\qquad \begin{aligned} \deg \alpha &= \max_{1 \leqslant i,j \leqslant n} \{\deg Q, \deg P_{ij}\}, \\ \operatorname{ht} \alpha &= \max_{1 \leqslant i,j \leqslant n} \{\operatorname{ht} Q, \operatorname{ht} P_{ij}\}, \\ \operatorname{size} \alpha &= \max\{\deg \alpha, \log \operatorname{ht} \alpha\}. \end{aligned}$$

Thus since elements P/Q in $\mathbf{Q}(x_1, \cdots, x_m)$ with P, Q relatively prime in I_K, have matrices $(\delta_{ij}P/Q)$ for multiplication, this definition of size in K agrees with the previous definition in $\mathbf{Q}(x_1, \cdots, x_m)$.

Absolute size. If $\alpha \in K$, define its *absolute size* to be the size of its minimal polynomial $P(z) \in I_K[z]$ when considered as a polynomial in the $m + 1$ variables $z, x_1, \cdots, x_m$. Define its *coefficient size* to be the maximum of the sizes of the coefficients of $P(z)$ in I_K.

REMARK 2.2. If α is nonzero in K and $M = (P_{ij}/Q)$ is the matrix used in (2.6), then the polynomial in z defined by $\det(zQI - QM)$ is a power (dividing n) of the minimal polynomial of α over I_K [29, p. 89]. The coefficient of z^k in that polynomial is a sum of $\binom{n}{k}$ terms, each of which is Q^k times an $(n-k) \times (n-k)$ minor of QM. Thus each coefficient has degree in $x_1, \cdots, x_n \leqslant n \deg \alpha$ and, by (2.10) of Lemma 2.3 below, size $\leqslant 2n$ size α + $n \log n$. Consequently by Lemma 2.1,

$$(2.7)\qquad \text{absolute size} \leqslant 3n \ \text{ size } + n(1 + \log n).$$

This inequality is remarkable inasmuch as the particular basis (2) of (2.1) is not at all involved. It is not clear whether one can obtain a similar inequality in the opposite direction of (2.7) which would also hold uniformly for all nonzero α in K. In fact, as we shall see shortly, it is not clear that absolute size obeys inequalities like those we established for "ordinary" size in the next lemma:

LEMMA 2.3. *If $\alpha_1, \cdots, \alpha_k$ are nonzero elements of $I_K = \mathbf{Z}[x_1, \cdots, x_m]$, then*

$$(2.8)\qquad \operatorname{ht}(\alpha_1 + \cdots + \alpha_k) \leqslant \operatorname{ht} \alpha_1 + \cdots + \operatorname{ht} \alpha_k,$$

$$(2.9)\qquad \deg(\alpha_1 + \cdots + \alpha_k) \leqslant \sum_{i=1}^{m} \max_j \deg_{x_i} \alpha_j \leqslant \deg \alpha_1 + \cdots + \deg \alpha_k,$$

$$(2.10)\qquad \operatorname{size}(\alpha_1 \cdots \alpha_k) \leqslant 2(\operatorname{size} \alpha_1 + \cdots + \operatorname{size} \alpha_k).$$

More generally if $\alpha_1, \cdots, \alpha_k$ *are any nonzero elements of* K, *then*

$$\text{size}(\alpha_1 + \cdots + \alpha_k) \leqslant 3(\text{size}\ \alpha_1 + \cdots + \text{size}\ \alpha_k) + \log k, \tag{2.11}$$

$$\text{size}(\alpha_1 \cdots \alpha_k) \leqslant 3(\text{size}\ \alpha_1 + \cdots + \text{size}\ \alpha_k) + k \log n. \tag{2.12}$$

PROOF. Assertions (2.8) and (2.9) are trivial. (2.10) follows immediately by induction on k using the facts that

$$\text{ht}(\alpha_1\alpha_2) \leqslant (1 + \deg_{x_1}\alpha_2) \cdots (1 + \deg_{x_m}\alpha_2)(\text{ht}\ \alpha_1)(\text{ht}\ \alpha_2),$$

$$\deg \alpha_1\alpha_2 = \deg \alpha_1 + \deg \alpha_2, \quad \text{and} \quad 1 + x \leqslant \exp x, \ \text{when} \ x \geqslant 0.$$

Assertion (2.11) follows upon putting all the α's over a common denominator and then using (2.10), (2.8) and (2.4). If all the numbers $\alpha_1, \cdots, \alpha_k$ have denominator 1, then matrix multiplication together with (2.8), (2.9) and (2.10) show that (2.12) holds with the number 3 replaced by 2. In the general case there may be some common factor between the entries in the matrix obtained by multiplying together the matrices (P_{ij}) and the product of the denominators. (2.4) then yields (2.12), since the product of the denominators satisfies the inequality (2.10).

REMARK 2.4. Since the definition of size is somewhat artificial, depending as it does on (1) and (2) of (2.1), it is relevant to ask whether absolute size might not also enjoy properties like those of size in Lemma 2.3. In case of a positive answer, we might well want to stick with the more intrinsic concept of "absolute size", which depends only on the choice of transcendence basis. Here we indicate the obstacle which prevented us from considering that course of action. If $f(x)$, the minimal polynomial for α over I_K, has degree r and $g(x)$, the minimal polynomial for β over I_K, has degree s, then since β is a root of $x^r f(\alpha\beta/x)$, $\alpha\beta$ is a root of the polynomial in t which is the resultant $R(x^r f(t/x), g(x))$ [12]. However Lemma 2.3 tells us only that the coefficient size of this polynomial is at most

$$2r \text{ size } g + 2s \text{ size } f + (r + s)\log(r + s). \tag{2.13}$$

As we expect from Lemma 2.1, the absolute size of $\alpha\beta$ may be even larger. Unfortunately induction on (2.13) does not yield a linear inequality in the sizes for arbitrary k as in (2.12). Similarly $\alpha + \beta$ satisfies $R(f(t - x), g(x))$, since β is a root of $f(\alpha + \beta - x)$, whose coefficient size is bounded by something of the general form of (2.13).

LEMMA 2.5. *There exists a constant* $C > 0$, *depending on* n, *such that for every nonzero* α *in* K,

$$(1/C)\ (\text{size}\ \alpha) - 1 \leqslant \text{size}\ 1/\alpha \leqslant C\ \text{size}\ \alpha + C.$$

PROOF. We need only establish the right-hand inequality since the other inequality follows from it with $1/\alpha$ replaced by α. Now the matrix for $1/\alpha$ is the inverse matrix, properly normalized, of the matrix for α. Let (P_{ij}/Q) be the matrix for multiplication by α as in (2.5). Then (QP^*_{ij}/Δ) is a matrix for multiplication by $1/\alpha$ where P^*_{ij} is the cofactor of P_{ij} in (P_{ij}) and $\Delta = \det(P_{ij})$, which are fixed polynomial functions depending on n in the entries of (P_{ij}), with

$$\text{size}\ P^*_{ij},\ \text{size}\ \Delta \leqslant 2n\ \text{size}\ \alpha + n \log n;$$

whereupon the result follows by (2.10) and (2.4).

LEMMA 2.6. *If $x_1, \cdots, x_m, y_1, \cdots, y_n$ and $x_1, \cdots, x_m, y'_1, \cdots, y'_n$ are two bases for K over $\mathbf{Q}$ in the sense of (2.1) and size′ denotes size with respect to the second basis, then there is a constant $C > 0$ depending only on the two bases such that for all nonzero α in K,*

$$C^{-1}(\text{size}\ \alpha) - 1 \leqslant \text{size}'\ \alpha \leqslant C\ \text{size}\ \alpha + C.$$

PROOF. Again we only need to prove, say, the inequality on the right. Let S_1 be the $n \times n$ matrix whose ith column consists of the coordinates of y'_i when expressed as a $\mathbf{Q}(x_1, \cdots, x_m)$-linear combination of $y_1, \cdots, y_n$, normalized as in (2.5). Similarly let S_2 be the normalized $n \times n$ matrix whose (i, j)th entry is the coordinate of y_i with respect to y'_j. If $M = (P_{ij}/Q)$ is the normalized matrix for multiplication by α with respect to $x_1, \cdots, x_m, y_1, \cdots, y_n$, then S_2MS_1 is the matrix for multiplication by α with respect to $x_1, \cdots, x_m, y'_1, \cdots, y'_n$. When S_2MS_1 is normalized as in (2.5), the denominator for S_2MS_1 is a factor of the product of the denominators for S_1, S_2 and M, and the numerators are factors of fixed I_K-linear combinations of the P_{ij}. The result now follows from Lemmas 2.3 and 2.1.

III. **Fields with transcendence type $\leqslant \tau$.** The idea of transcendence type originated with Lang [13], who first proved Lemma 3.7 below. If $K \subseteq \mathbf{C}$ is a field with transcendence basis $x_1, \cdots, x_m$, over $\mathbf{Q}$, then we say that K has *transcendence type* $\leqslant \tau$ if there is a constant $C_0 > 0$ such that for every nonzero $\alpha \in \mathbf{Q}(x_1, \cdots, x_m)$,

$$-(\text{size}\ \alpha)^\tau \leqslant C_0 \log |\alpha|. \tag{3.1}$$

We fix K, τ, C_0 and $\{x_1, \cdots, x_m\}$ for the rest of this section.

REMARK 3.1. In the above situation the Dirichlet box principle implies that $\tau \geqslant m + 1$. However there are very few cases where it has been shown that this lower bound is nearly an upper bound.

REMARK 3.2. Any algebraic number field has transcendence type $\leqslant 1$, since for any nonzero $\alpha = r/s$ in $\mathbf{Q}$, $s > 0$,

$$- (\text{size } \alpha) = - \log s = \log 1/s \leqslant \log |r/s|.$$

REMARK 3.3. Feldman [8] has shown that any finite extension of $\mathbf{Q}(\pi)$ has transcendence type $\leqslant 2 + \epsilon$, for every positive ϵ (2 is best possible). Relatively large classes of numbers are known [7], [9] to generate fields of transcendence degree one and transcendence type $\leqslant 4 + \epsilon$, for every positive ϵ.

The following lemma generalizes a well-known fact about algebraic numbers.

LEMMA 3.4. *There is a constant* $C_1 > 0$ *depending only on* $\{x_1, \cdots, x_m\}$ *such that for every complex number* α *algebraic over* $\mathbf{Q}(x_1, \cdots, x_m)$,

$$|\alpha| < 1 + \exp C_1 \sigma(\alpha)^\tau \tag{3.2}$$

where $\sigma(\alpha)$ *denotes the coefficient size of* α *over* $I_K = \mathbf{Z}[x_1, \cdots, x_m]$.

PROOF. We may assume without loss of generality that $|\alpha| > 1$. Let

$$P(z) = a_0 z^d + \cdots + a_d$$

be the minimal polynomial for α over I_K. Since $P(\alpha) = 0$, $|a_0 \alpha^d| = |a_1 \alpha^{d-1} + \cdots + a_d|$. If we set $H = \max_{1 \leqslant i \leqslant d} |a_i|$, then

$$|a_0 \alpha^d| \leqslant H(|\alpha|^{d-1} + \cdots + 1) = H(|\alpha|^d - 1)/(|\alpha| - 1) < H|\alpha|^d/(|\alpha| - 1),$$

and

$$|\alpha| - 1 < H/|a_0|.$$

(3.1) gives an upper bound of $\exp(C_0 \sigma(\alpha)^\tau)$ on $1/|a_0|$. If we set $\delta = \max\{\deg a_i\}_{i=1}^d$, where deg denotes as in (2.3) the sum of the degrees with respect to each x_i, then we see that each a_i involves at most $\exp(\delta P)$ summands, each of which has absolute value at most $\operatorname{ht} P \prod_{i=1}^m \max\{1, |x_i|\}^{\delta P}$. Taking $C_1 = 2 + C_0 + \sum_{|x_i| > 1} \log|x_i|$ gives the result announced.

COROLLARY 3.5. *In the above situation,* $-1 - C_1 \sigma(\alpha)^\tau < \log|\alpha| < 1 + C_1 \sigma(\alpha)^\tau$.

PROOF. The right-hand inequality follows directly from Lemma 3.3 and the fact that $1 + e^x \leqslant e^{1+x}$ when $x \geqslant 0$. The left inequality follows from the right one applied to $1/\alpha$.

COROLLARY 3.6. *If K is a subfield of $\mathbf{C}$ finitely generated over $\mathbf{Q}$, with transcendence basis $\{x_1, \cdots, x_m\}$, then there is a constant C such that, for every $\alpha \in \mathbf{C}$ which is integral over K,* $|\alpha| < 1 + \exp C$ size α.

PROOF. $|a_0| = 1$ in the proof of Lemma 3.4.

LEMMA 3.7. *Then for each extension of $\{x_1, \cdots, x_m\}$ to a basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ as in (2.1), there is a constant $C_K > 0$ such that for all nonzero α in K*

$$- C_K - C_K(\text{size } \alpha)^\tau \leq \log |\alpha|. \tag{3.3}$$

PROOF. Take, for example, $C_K = \max\{1, (6n)^\tau(C_1), -\log|\beta|\}$ where β is allowed to run over all the finitely many nonzero elements of K with absolute size less than $2n(1 + \log n)$. Then when absolute size $\alpha \geq 2n(1 + \log n)$, by (2.7) $3n$ size $\alpha \geq$ absolute size $\alpha - n(1 + \log n) \geq$ (absolute size $\alpha)/2$. The claim is then established by Corollary 3.5.

We shall reserve the symbol C_K for the constant in Lemma 3.7 with respect to the basis fixed in (2.1).

If $K = \mathbf{Q}(x_1, \cdots, x_m)$ is a field having transcendence type $\leq \tau$ with respect to $x_1, \cdots, x_m$, then many familiar results over $\mathbf{Z}$ can be generalized to I_K. We shall investigate a few of them here to familiarize the reader with such techniques and to emphasize the extent to which $\mathbf{Z}$ and I_K are analogous.

However we do not intend here to establish the sharpest possible results. Rather we shall prefer simple expressions of generally correct shape to cumbersome, more exact ones. Only the next four results are necessary to understand the rest of the paper, the first of which is standard and has nothing to do with I_K.

For any $f(z) \in \mathbf{C}[z]$, define $\|f\|$ to be the largest absolute value of any of the coefficients of f.

LEMMA 3.8. *Let $f(z), g(z) \in \mathbf{C}[z]$ have resultant $R(f, g)$. Then for any complex number ω,*

$$|R(f, g)| \leq \max\{|f(\omega)|, |g(\omega)|\} \|f\|^{s-1} \|g\|^{r-1}(s\|f\| + r\|g\|)(1 + r)^{s/2}(1 + s)^{r/2},$$

where f and g have degrees r and s (in z), respectively.

PROOF. Let $f(z) = a_0z^r + \cdots + a_r$, $g(z) = b_0z^s + \cdots + b_s$. Then $R(f, g)$ is the determinant of the matrix in Figure 1, where all entries outside the two rectangles are zero. By adding ω^{r+s-i} times the ith column, $1 \leq i < r + s$, to the last column, $R(f, g)$ is also seen to be equal to the determinant of the matrix obtained by replacing the last column of Figure 1 by the transpose of the vector

$$(3.5) \qquad (\omega^{s-1}f(\omega), \omega^{s-2}f(\omega), \cdots, f(\omega), \omega^{r-1}g(\omega), \cdots, g(\omega)).$$

When $|\omega| \leqslant 1$, the inequality follows by expanding the determinant along the last column and applying Hadamard's inequality to the cofactors. When $|\omega| > 1$, add instead ω^{1-i} times the ith column, $1 < i \leqslant r+s$, to the first column to get a matrix with the polynomials evaluated at ω in the first column.

$$\begin{array}{l}
\left|\begin{array}{cccccccc}
a_0 & a_1 & \cdots & a_r & & & & \\
 & \ddots & & & \ddots & & & \\
 & & a_0 & a_1 & \cdots & & & a_r \\
b_0 & \cdots & b_{s-1} & b_s & & & & \\
 & \ddots & & & \ddots & & & \\
 & & & b_0 & \cdots & b_{s-1} & & b_s
\end{array}\right|
\begin{array}{l} s \text{ times} \\ \\ \\ r \text{ times} \\ \\ \\ \end{array}
\end{array}$$

Figure 1

COROLLARY 3.9. *Let $f(z), F(z) \in I_K[z]$ with degrees r and R (in z), respectively, and with coefficients with maximal sizes s and S, respectively. Then $f(z)$ and $F(z)$ have a common factor involving z if and only if, for some complex number ω,*

$$(3.6) \qquad \max\{|f(\omega)|, |F(\omega)|\} < \exp\{-C_1(2(rS+Rs) + (r+R)\log(r+R))^\tau\}.$$

PROOF. If $f(z)$ and $F(z)$ have a common factor, then they have a common root ω in $\mathbf{C}$.

It is standard [12] that if the resultant $R(f, F)$ vanishes, then $f(z)$ and $F(z)$ have a common zero. We saw in the proof of Lemma 3.4 that $\|f\| \leqslant \exp C_1 s$ and $\|F\| \leqslant \exp C_1 S$. Moreover $(r+1)^{R/2}(R+1)^{r/2}(r+R) < (r+R)^{r+R}$. Therefore by (3.6) and Lemma 3.8,

$$|R(f, F)| < \exp\{-C_0((rS+Rs) + (r+R)\log(r+R))^\tau\}.$$

But Lemma 2.3 shows that size $R(f, F) \leqslant 2(rS+Rs) + (r+R)\log(r+R)$, which contradicts (3.1) unless $R(f, F)$ is zero.

With that we have established what is sometimes [13] called a "Liouville estimate":

COROLLARY 3.10. *Let $P(z) \in I_K[z]$ have degree r in z and have coefficients with maximal size s. Let $\alpha \in \mathbf{C}$ be algebraic over $\mathbf{Q}(x_1, \cdots, x_m)$ with minimal polynomial $Q(z) \in I_K[z]$ with degree R in z and with coefficients of maximum size S. Then either*

$$P(\alpha) = 0 \quad or \quad |P(\alpha)| \geqslant \exp\{-(C_1(2(rS+Rs) + (r+R)\log(r+R)))^\tau\}.$$

PROOF. Apply Lemma 3.9 to $P(z)$ and the minimal polynomial of α.

LEMMA 3.11. *There is a constant $C_2 > 0$ depending on $\{x_1, \cdots, x_m\}$ such that the following holds: Let $P(z) \in I_K[z]$ have degree d in z and coefficients with maximal size s. Assume that $P(z)$ has distinct roots $\xi_1, \cdots, \xi_d$ in $\mathbf{C}$ and let ω in $\mathbf{C}$ be arbitrary. Then*

$$\min_{1 \leqslant i \leqslant d} |\omega - \xi_i| \leqslant |P(\omega)| \exp C_2(ds + d \log d)^\tau.$$

PROOF. Order the ξ_i so that $|\omega - \xi_1| \leqslant \cdots \leqslant |\omega - \xi_d|$. Then $|\xi_1 - \xi_i| \leqslant |\xi_1 - \omega| + |\omega - \xi_i| \leqslant 2|\omega - \xi_i|$. Consequently

$$\begin{aligned} |P(\omega)| &= |a_d||\omega - \xi_1| \cdots |\omega - \xi_d| \\ &\geqslant |a_d||\omega - \xi_1||\xi_1 - \xi_2| \cdots |\xi_1 - \xi_d|/2^{d-1} \\ &= |P'(\xi_1)|\,|\omega - \xi_1|/2^{d-1}. \end{aligned}$$

Since $P(z)$ has no repeated roots, $P'(\xi_1) \neq 0$, and we can apply Corollary 3.10 to $P'(z)$, which has size at most $s + \log d$.

THEOREM 3.12. *Let $\lambda, \mu \colon \mathbf{N} \times \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}_{>0}$.*

(a) *If ω is a complex number and if $|\omega - \alpha| > \exp - \lambda(d, s)$ for all complex numbers algebraic over $\mathbf{Q}(x_1, \cdots, x_m)$ of degree d and coefficient size s, then for all nonconstant polynomials $P(z) \in I_K[z]$ with no repeated roots having degree d in z and coefficients of maximum size s,*

$$|P(\omega)| > \exp\{- \lambda(d, d + 2s) + C_2(ds + d \log d)^\tau\}.$$

(b) *If, on the other hand, $|P(\omega)| > \exp - \mu(d, s)$ for all nonconstant irreducible polynomials $P(z)$ in $I_K[z]$ of degree d and coefficients having maximal size s, then there is a constant $C_3 > 0$ so that*

$$|\omega - \alpha| > \exp\{- \mu(d, s) + C_3 ds^\tau\}$$

for all numbers α which are algebraic over $\mathbf{Q}(x_1, \cdots, x_m)$ of degree d and absolute size s.

PROOF. (a) follows immediately from Lemma 3.11.

(b) Let $\alpha \in C$ be algebraic over $\mathbf{Q}(x_1, \cdots, x_m)$ with minimal polynomial $P(z) = a_0 x^d + \cdots + a_d$, $s = \max\{d, \text{size } a_i\}$. We know from Corollary 3.5 that the roots of P are bounded in absolute value by $\exp\{(1 + C_1)s^\tau\}$. (Note that by taking absolute size, we guarantee that $s \geqslant 1$.) Thus

$$\begin{aligned} |P(\omega)| &= |a_d|\,|\omega - \alpha_1| \cdots |\omega - \alpha_d| \\ &\leqslant |a_d|\,|\omega - \alpha|(\max\{2, |\omega|\})^{d-1} \exp\{(d - 1)(1 + C_1)s^\tau\}. \end{aligned}$$

Take $C_3 = 1 + C_1 + \max\{1, \log|\omega|\}$. Then the conclusion holds since $|a_d| \leqslant \exp\{(1 + C_1)s^\tau\}$.

IV. **Transcendence and approximating sequences.** The purpose of this section is to generalize Gelfond's crucial characterization [11] of algebraic numbers in terms of the approximations they admit to include numbers algebraic over a field of transcendence type $\leqslant \tau$. Fix $K = \mathbf{Q}(x_1, \cdots, x_m)$ with transcendence type $\leqslant \tau$ with respect to the transcendence basis $x_1, \cdots, x_m$ for this section. Our first lemma shows that if a polynomial over I_K is small, in an appropriate sense, at a point in $\mathbf{C}$, then it is small because some irreducible factor over I_K is small there:

LEMMA 4.1. *There is a constant $C_4 > 0$ depending on $\{x_1, \cdots, x_m\}$ such that if $P(z) \in I_K[z]$, $C > C_4$, and $\omega \in \mathbf{C}$ satisfy the inequality*

$$|P(\omega)| < \exp - C(\delta\sigma)^\tau$$

where $\delta = \deg_z P(z)$ and the sizes of the coefficients of P are at most $\sigma > 0$, then there is a divisor $Q(z)$ of $P(z)$ which is a power of an irreducible polynomial over I_K such that

$$|Q(\omega)| < \exp(C_4 - C)(\delta\sigma)^\tau.$$

PROOF. The proof is, mutatis mutandis, Gelfond's and comes in two stages:

(a) If $P(z)$ is itself a power of an irreducible polynomial over I_K, there is nothing more to show. So assume $f(z), F(z) \in I_K[z]$ are relatively prime divisors of $P(z)$. By Lemma 2.1, we know that the coefficients of $f(z)$ and $F(z)$ have size at most 2σ. By Lemma 3.9, we know that

$$\max\{|f(\omega)|, |F(\omega)|\} \geqslant \exp\{- C_1(8\delta\sigma + \delta \log \delta)^\tau\}. \tag{4.3}$$

(b) In the general case we factor $P(z)$ over I_K into a product of powers of distinct irreducible factors $P(z) = P_1(z) \cdots P_s(z)$ with $|P_1(\omega)| \leqslant |P_2(\omega)| \leqslant \cdots \leqslant |P_s(\omega)|$. Interpreting the empty product to be the number 1, we see that the inequality

$$|P_1(\omega) \cdots P_k(\omega)| > |P_{k+1}(\omega) \cdots P_s(\omega)| \tag{4.4}$$

holds for $k = 0$, but not for $k = s$. Let l be the least natural number for which (4.4) fails. If $l = 1$, then

$$\begin{aligned} |P_1(\omega)| &= |P(\omega)|/|P_2(\omega) \cdots P_s(\omega)| \\ &< |P(\omega)| \exp C_1 9^\tau(\delta\sigma)^\tau, \end{aligned}$$

by (4.3). If $l > 1$, then

$$|P_1(a) \cdots P_{l-1}(\omega)| > |P_l(\omega)|\,|P_{l+1}(\omega) \cdots P_s(\omega)|,$$

and

$$|P_1(\omega) \cdots P_{l-1}(\omega)|\,|P_l(\omega)| \leqslant |P_{l+1}(\omega) \cdots P_s(\omega)|.$$

We apply (4.3) to conclude that

$$|P_1(\omega) \cdots P_{l-1}(\omega)| > \exp\{-9^\tau C_1(\delta\sigma)^\tau\}, \tag{4.5}$$

$$|P_{l+1}(\omega) \cdots P_s(\omega)| > \exp\{-9^\tau C_1(\delta\sigma)^\tau\}. \tag{4.6}$$

Dividing $|P(\omega)|$ by the left-hand side of (4.5) and (4.6), we see that

$$|P_l(\omega)| < |P(\omega)| \exp\{2C_1 9^\tau(\delta\sigma)^\tau\}.$$

The result is established if we take $C_4 = 2C_1 9^\tau$.

REMARK 4.2. One can, by requiring $C > 3C_1 9^\tau$, force l to be 1 and obtain $|Q(\omega)| < \exp(9^\tau C_1 - C)(\delta\sigma)^\tau$. (Apply (a) to $P_1(\omega)$ and $P_l(\omega)$.)

THEOREM 4.3. *Let $a > 1$. Then there is a constant $C_5 > 0$ depending on a and $\{x_1, \cdots, x_m\}$ such that any complex number ω is algebraic over K if and only if there are nondecreasing functions δ, σ: $\mathbf{N} \longrightarrow \mathbf{R}$ with positive range, σ unbounded,*

$$\delta(n+1) \leqslant a\delta(n), \tag{4.7}$$

$$\sigma(n+1) \leqslant a\sigma(n), \tag{4.8}$$

for all positive integers n, and furthermore there is a sequence of nonzero polynomials $P_n(z)$ over I_K for which

$$\deg_z P_n(z) \leqslant \delta(n), \tag{4.9}$$

$$\text{absolute size } P_n(z) \leqslant \sigma(n), \tag{4.10}$$

and

$$\log|P_n(\omega)| \leqslant -C_5(\delta(n)\sigma(n))^\tau. \tag{4.11}$$

PROOF. If ω is algebraic over K, always take $P_n(z)$ to be the minimal polynomial for ω over I_K, $\delta(n)$ to be the degree of ω over $\mathbf{Q}(x_1, \cdots, x_m)$, $\sigma(n) = n +$ absolute size of ω, and $a = \sigma(2)/\sigma(1)$.

The converse will be established in three steps, which we outline here before proceeding. Lemma 4.1 will be applied to each $P_n(z)$ to obtain a sequence of polynomials in $I_K[z]$ which are powers of irreducible polynomials in $I_K[z]$, but which satisfy an only slightly weaker inequality than (4.11). Then Lemma 3.9 shows that the underlying small irreducible factor of $P_n(z)$ is the same for each large n and $n+1$, and hence for all large n. This conclusion, however, since $\delta(n)\sigma(n)$ tends to infinity with n, is tenable only if the value of that irreducible polynomial is zero at ω. Hence the underlying irreducible polynomial is the minimal polynomial for ω over I_K and, for each n large, $P_n(\omega) = 0$.

Step I. Let $C_5 = 3(9a)^\tau C_1$. Then by Lemma 4.1, since $C_4 = 2C_1 9^\tau$, we obtain a sequence $Q_n(z)$ of powers of irreducible polynomials in $I_K[z]$ satisfying

$$|Q_n(\omega)| \leqslant -(C_5/3)(\delta(n)\sigma(n))^\tau. \tag{4.12}$$

Moreover by Lemma 2.1,

$$\text{coefficient size } Q_n(z) \leqslant 2\sigma(n), \qquad \deg_z Q_n(z) \leqslant \delta(n).$$

Step II. By Lemma 3.9, we know that if the underlying irreducible polynomials for $Q_n(z)$ and $Q_{n+1}(z)$ were different, then

$$\begin{aligned} \max\{|Q_n(\omega)|, |Q_{n+1}(\omega)|\} &\geqslant \exp\{-C_1((2(\delta(n)2\sigma(n+1) + \delta(n+1)2\sigma(n)) \\ &\qquad + (\delta(n) + \delta(n+1))\log(\delta(n) + \delta(n+1)))^\tau\} \\ &\geqslant \exp\{-C_1(8a\delta(n)\sigma(n) + (a+1)\delta(n)\log((a+1)\delta(n)))^\tau\} \\ &> \exp\{-C_1(9a)^\tau(\delta(n)\sigma(n))^\tau\}, \end{aligned}$$

when $\sigma(n) > 2\log((a+1)\sigma(n))$, say. However since $C_5 = 3(9a)^\tau C_1$, we know that this inequality holds for no sufficiently large n.

Step III. Let $Q(z) \in I_K[z]$ be the underlying irreducible polynomial for all $Q_n(z)$, n large. Let us say that for large n, $Q_n(z) = Q(z)^{r_n}$. Then since $\tau \geqslant 1$, (4.12) shows that for large n

$$|Q(\omega)|^{r_n} = |Q_n(\omega)| \leqslant \exp - (C_5/2)\delta(n)\sigma(n).$$

Since $r_n < \delta(n)$, we know then also that for all large n,

$$|Q(\omega)| < \exp - (C_5/2)\sigma(n).$$

Because $\sigma(n) \uparrow \infty$, $Q(\omega)$ must be zero. Consequently ω is algebraic over I_K with minimal polynomial $Q(z)$.

REMARK 4.4. It is not difficult to choose a slightly larger C_5 and show that *all* $P_n(\omega) = 0$.

REMARK 4.5. One can generalize the theorem by letting a: $\mathbf{N} \longrightarrow \mathbf{R}$ with $a(n) \geqslant 1$ in (4.7) and (4.8). Then C_5 is also a function of N.

REMARK 4.6. If the constant C_0 from the definition of transcendence type is known, then an upper bound for the constant C_5 of Theorem 4.3 can be given explicitly from the definition of C_1, remembering from (3.1) that $|x_i| \leqslant \exp C_0$.

One can also prove a result analogous to Theorem 4.3, but allowing approximation by complex numbers algebraic over I_K:

THEOREM 4.7. *Let* $a > 1$. *The complex number* ω *is algebraic over* K. *if and only if there are nondecreasing positive functions* δ, σ: $\mathbf{N} \longrightarrow \mathbf{R}$ *with* σ *unbounded and* $\sigma(n+1) \leqslant a\sigma(n), \delta(n+1) \leqslant a\delta(n)$ *for all positive integers* n *and a sequence* $\{\alpha_n\}$ *of complex numbers algebraic over* I_K *with*

$$\deg \alpha_n \leqslant \delta(n), \quad \text{absolute size } \alpha_n \leqslant \sigma(n),$$

and

$$|\omega - \alpha_n| \leqslant \exp\{-(C_3 + C_5)(\delta(n)\sigma(n))^\tau\}.$$

PROOF. If ω is algebraic, take each α_n to be $\omega, \delta(n) = \deg \omega, \sigma(n) = n +$ absolute size ω, and $a = \sigma(2)/\sigma(1)$.

For the converse, (b) of Theorem 3.12 shows that there is for each n an irreducible polynomial $P_n(z) \in I_K[z]$ with

$$\deg_z P_n(z) \leqslant \delta(n), \quad \text{absolute size } P_n(z) \leqslant \sigma(n),$$

and

$$|P_n(\omega)| \leqslant \exp\{-C_5(\delta(n)\sigma(n))^\tau\}.$$

We can now apply Theorem 4.3 to the sequence $\{P_n(z)\}$.

V. **Basic lemmas of Siegel and Tijdeman.** The first basic result, which is well known, treats the solution of systems of linear diophantine equations. Such lemmas based on the Dirichlet box principle are traditionally called "Siegel's Lemma", since he introduced this important technique into transcendence proofs.

LEMMA 5.1. *Let* R *and* S *be positive integers,* $R < S$ *and let* $a_{ij} \in \mathbf{Z}$, $1 \leqslant i \leqslant R$, $1 \leqslant j \leqslant S$, *have absolute values at most* $A \geqslant 1$. *Then the system of equations*

$$\begin{aligned} a_{11}z_1 + \cdots + a_{1S}z_S &= 0 \\ &\vdots \\ a_{R1}z_1 + \cdots + a_{RS}z_S &= 0 \end{aligned} \tag{5.1}$$

has a solution in integers z_i, not all zero, with

$$|z_i| \leqslant (SA)^{R/(S-R)}. \tag{5.2}$$

PROOF. We view evaluation of the left side of (5.1) as a linear function $L\colon \mathbf{Z}^{(S)} \longrightarrow \mathbf{Z}^{(R)}$. The strategy of the proof is to show by counting that L must take on the same value at distinct elements of $\mathbf{Z}^{(S)}$ whose coordinates are *nonnegative* integers satisfying (5.2). Their difference still satisfies (5.2) and, by linearity, also (5.1). To these ends, we need an inequality:

Let Z be the integral part of $(SA)^{R/(S-R)}$. Then $Z + 1 > (SA)^{R/(S-R)}$. Then $(Z + 1)^{S-R} > (SA)^R$, and

$$(Z+1)^S > (Z+1)^R (SA)^R > (SAZ+1)^R. \tag{5.3}$$

There are $(Z + 1)^S$ elements of $\mathbf{Z}^{(S)}$ whose coordinates lie in $[0, Z]$. Their images under L in $\mathbf{Z}^{(R)}$ have jth coordinates in $[-N_jAZ, (S - N_j)AZ]$, where N_j is the number of negative a_{ji}. Thus the images are in a subset of $\mathbf{Z}^{(R)}$ with $(SAZ + 1)^R$ elements. By (5.3), the box principle now implies that L takes on the same value at distinct points as desired.

We now extend Siegel's Lemma to apply to systems of linear equations with coefficients from $I_K = \mathbf{Z}[x_1, \cdots, x_m]$.

LEMMA 5.2. *Let R and S be positive integers, $2^mR < S$ and $a_{ij} \in I_K$, $1 \leqslant i \leqslant R$, $1 \leqslant j \leqslant S$ satisfy*

$$\deg a_{ij} \leqslant \delta, \tag{5.3}$$

$$\operatorname{ht} a_{ij} \leqslant A, \tag{5.4}$$

where $A \geqslant 1$. Then the system of equations

$$\begin{aligned} a_{11}z_1 + \cdots + a_{1S}z_S &= 0 \\ &\;\vdots \\ a_{R1}z_1 + \cdots + a_{RS}z_S &= 0 \end{aligned} \tag{5.5}$$

has a nonzero solution in $I_K^{(S)}$ satisfying

$$\deg z_k \leqslant \delta m, \quad \operatorname{ht} z_k \leqslant ((1+\delta)^{2m}SA)^{2^mR/(S-2^mR)}.$$

PROOF. Let each z_k be considered as a polynomial with undetermined coefficients in $x_1, \cdots, x_m$ having degree in each x_i equal to δ. We multiply these expressions out in (5.5) and collect coefficients of the monomials in $x_1, \cdots, x_m$. By setting each of these coefficients equal to zero, we obtain at most $R \cdot (1 + 2\delta)^m$ linear equations in $S \cdot (1 + \delta)^m$ unknowns, where the integer coefficients of these new linear equations are at most $A(1 + \delta)^m$ in absolute value. Lemma 5.1 insures the existence of nonzero z_k in I_K satisfying

$$\operatorname{ht} z_k \leqslant (S(1+\delta)^{2m}A)^{R(1+2\delta)^m/(S(1+\delta)^m - R(1+2\delta)^m)} \quad \text{and} \quad \deg z_k \leqslant \delta m.$$

Since $1 + 2\delta < 2(1+\delta)$,

$$R(1+2\delta)^m/(S(1+\delta)^m - R(1+2\delta)^m) \leqslant 2^m R/(S - 2^m R).$$

The following elegant theorem of Tijdeman, which improves on an earlier result of Gelfond [11, pp. 140–141], allows the application of the main theorem of §IV. We quote only the result, since the proof (see [22], [26]) would lead us too far afield.

THEOREM 5.3. *Let* $\alpha_j, A_{ij} \in \mathbf{C}$ *for* $1 \leqslant j \leqslant r$, $0 \leqslant i \leqslant s-1$ *and let* $\Delta = \max_j |\alpha_j|$. *Suppose that*

$$F(z) = \sum_{i=0}^{s} \sum_{j=1}^{r} A_{ij} z^i \exp(\alpha_j z)$$

is not identically zero. Then the number of zeros of F, *counting multiplicities, in the disc* $\{z \in \mathbf{C}: |z| \leqslant R\}$ *is at most* $3rs + 4R\Delta$.

VI. Proofs of the theorems.

PROOF OF THEOREM A. The object of the proof is to derive a contradiction from the supposition that the numbers $\alpha_1, \cdots, \alpha_M; \beta_1, \cdots, \beta_N$ present a counterexample to the theorem. If we do have such a counterexample, then the numbers α_i, β_j; $\exp(\alpha_i\beta_j)$, $i = 1, \cdots, M$; $j = 1, \cdots, N$, all lie in a field L of transcendence degree one over K of the following type:

(6.1)
(1) $L = \mathbf{Q}(x_0, x_1, \cdots, x_m, y)$,
(2) $\{x_1, \cdots, x_m\}$ is a transcendence basis for K,
(3) $\{x_0, x_1, \cdots, x_m\}$ is a transcendence basis for L,
(4) L is Galois over $\mathbf{Q}(x_0, x_1, \cdots, x_m)$ of degree l.

We let of course $I_L = \mathbf{Z}[x_0, x_1, \cdots, x_m]$. Let $A_i, B_j, C_{ij} \in I_L$ be denominators for $\alpha_i, \beta_j, \exp(\alpha_i\beta_j)$, respectively with respect to $(x_0, x_1, \cdots, x_m, 1, y, \cdots, y^{l-1})$, as preceding (2.6), $i = 1, \cdots, M$; $j = 1, \cdots, N$. Finally set $A = A_1 \cdots A_M$, $B = B_1 \cdots B_N$, and $C = \Pi C_{ij}$.

Let U be an integral parameter, which we shall presently let tend to infinity and set

$$p(U) = [U^{(M+N)/N}/\log U]. \tag{6.2}$$

Consider the general exponential polynomial $f_U(z) = \Sigma A_{\kappa\iota} Z^\kappa \exp(\alpha_\iota z)$ where the $A_{\kappa\iota} \in I_L$ will be chosen later, $\iota = (i_1, \cdots, i_M)$, $\alpha_\iota = i_1\alpha_1 + i_2\alpha_2 + \cdots + i_M\alpha_M$, and the sum is taken over all κ, ι with $0 \leqslant \kappa < p(U)$ and $0 \leqslant i_1, i_2, \cdots, i_M < U$. We shall select the coefficients $A_{\kappa\iota}$ not all zero, so that

f_U has a zero of order at least $p(U)$ at every lattice point

$$j = j_1\beta_1 + \cdots + j_N\beta_N \tag{6.3}$$

with $0 \leqslant j_1, \cdots, j_N < [(U^M/2^{m+2}l^2)^{1/N}]$.

By Leibnitz's rule,

$$\left(\frac{d}{dz}\right)^p (e^{az}z^i) = e^{az} \sum_{r=0}^{p} \binom{p}{r} \frac{i!}{(i-r)!}\, a^{p-r}z^{i-r}, \tag{6.4}$$

for any $a \in \mathbf{C}$ and $I \in \mathbf{Z}$. We know that when j is given by (6.3), each

$$(AB)^{p(U)}C^{[U^{1+M/N}]}\alpha_\iota^\sigma j^\tau \exp(\alpha_\iota j), \tag{6.5}$$

$0 \leqslant \sigma, \tau < p(U)$, has matrix for multiplication with entries from I_L. Moreover (2.12) implies that (6.5) has

$$\text{size} \ll p(U) \log U + U^{(M+N)/N} \ll U^{(M+N)/N} \tag{6.6}$$

with respect to $(x_0, x_1, \cdots, x_m, 1, y, \cdots, y^{l-1})$. (Here as below, the positive constants implicit in our use of $\ll$ are independent of U.)

In the application of (6.4) which we are considering,

$$\binom{p}{r} \leqslant p^p \leqslant p(U)^{p(U)} \quad \text{and} \quad i!/(i-r)! \leqslant p(U)^{p(U)}.$$

Thus, in view of (6.4) and (6.6), when we evaluate

$$(AB)^{p(U)}C^{U[U^{M/N}]}\left(\frac{d}{dz}\right)^p (z^\kappa \exp(\alpha_\iota z)), \tag{6.7}$$

$0 \leqslant p < p(U)$, at the lattice points given in (6.3), we obtain elements of L whose multiplication matrices have entries from I_L with

$$\text{sizes} \ll U^{(M+N)/N}. \tag{6.8}$$

Multiplying (6.5) by $A_{\kappa\iota}$ and setting the sums over κ and ι equal to zero for each $0 \leqslant p < p(U)$ insures that f_U has zeros of order at least $p(U)$ at the points (6.3). Now in order to set the sums equal to zero, it suffices to do it to each of the l^2 entries (in I_L) of the matrix for multiplication by the sum. For zero is the only element in L with multiplication matrix having all coordinates zero. In this manner we get a system of homogeneous linear equations in the unknowns $A_{\kappa\iota}$ whose coefficients are entries of the matrices for the elements of (6.7) evaluated at points of (6.3). They thus lie in I_L and have

$$\text{size} \ll U^{(M+N)/N}. \tag{6.9}$$

But the number of unknowns $= U^M p(U)$ and the number of equations $\leqslant p(U) \times U^M/2^{m+2}$. Thus by Lemma 5.2, we know that there is a nontrivial solution for the $A_{\kappa\iota}$ in I_L such that each $A_{\kappa\iota}$ satisfies

$$\text{size } A_{\kappa\iota} \ll U^{(M+N)/N}. \tag{6.10}$$

Since at least one of the $A_{\kappa\iota}$ is nonzero and $\alpha_1, \cdots, \alpha_M$ are linearly independent over Q, f_U is not identically zero. But since $\beta_1, \cdots, \beta_N$ are linearly independent over Q as well, they generate a lattice of dimension N. However Lemma 5.4 shows that when $U \geqslant U_0$, $f_U(z)$ does not vanish to an order of $p(U)$ at some lattice point

$$\beta_U = j_1\beta_1 + \cdots + j_N\beta_N, \tag{6.11}$$

where $0 \leqslant j_1, \cdots, j_N \leqslant 2U^{M/N}$. Let us say

$$A_U = f_U^{p_U}(\beta_U) \neq 0, \qquad 0 \leqslant p_U < p(U).$$

We shall apply the Cauchy integral formula to estimate the absolute value of A_U. Since f_U has a zero of multiplicity at least $p(U)$ at each point of (6.3), when $U \geqslant U_1 \geqslant U_0$, we can write

$$A_U = \frac{p_U!}{-4\pi^2} \int_\Gamma \frac{dz}{(z-\beta_U)^{p_U+1}} \int_{\Gamma^1} \prod \left(\frac{z-j}{\omega-j}\right)^{p(U)} \frac{f_U(\omega)}{\omega - z}\, d\omega, \tag{6.12}$$

where the product is taken over all the lattice points j of (6.3) the curve Γ is the circle $|z| = U^{(M+1)/N}$ and the curve Γ^1 is the circle $|\omega| = U^M$. From (6.9) and the definition of f_U, we see that for ω on Γ^1, $\log|f_U(\omega)| \ll U^{M+1}$. Moreover when $U \geqslant U_2 \geqslant U_1$ and z is on Γ, $|z - \beta_U| > 1$ and the absolute value of each $(z-j)/(\omega - j)$ is at most, say, $4U^{-1/2}$. Thus, keeping (6.10) in mind, we see from (6.12) that when $U \geqslant U_3 \geqslant U_2$,

$$\begin{aligned} \log|A_U| &\ll p(U)^2 - p(U) \cdot (U^M/2^{m+2}l^2)(\log U) + U^{M+1} \\ &\ll -p(U)U^M \log U. \end{aligned}$$

We can therefore conclude that when $U \geqslant U_4 \geqslant U_3$,

$$0 \neq (AB)^{p(U)} C^{2U[U^{M/N}]} A_U = B_U$$

has

$$\log|B_U| \ll -p(U)U^M \log U, \tag{6.13}$$

and by (2.12),

$$\text{size } B_U \ll U^{1+M/N}. \tag{6.14}$$

Consequently every conjugate B_U^σ of B_U over $Q(x_0, x_1, \cdots, x_m)$ has by (2.7) and (6.14)

$$\text{(6.15)} \qquad \text{absolute size } B_U^\sigma = \text{absolute size } B_U \ll U^{1+M/N}.$$

Moreover, the matrix for multiplication by B_U has coefficients from I_L. But B_U itself is a root of the characteristic polynomial for that matrix by the Cayley-Hamilton theorem. Thus B_U and each of its conjugates is integral over I_L. Thus by Corollary 3.6,

$$\text{(6.16)} \qquad \log|B_U^\sigma| \ll U^{1+M/N}.$$

Multiplying B_U together with all its conjugate gives $0 \neq C_U = \Pi B_U^\sigma \in I_L$ and by (6.12) and (6.15)

$$\log|C_U| \ll -p(U)U^M \log U + U^{1+M/N} \ll -p(U)U^M \log U.$$

Moreover since C_U is the norm of B_U, size $C_U \leqslant$ absolute size B_U. Thus by (6.15), size $C_U \ll U^{1+M/N}$.

Now since $C_U \in I_L$, we know that $C_U = P_U(x_0), P_U(z) \in I_K[z]$ with $I_K = Z[x_1, \cdots, x_m]$,

$$\deg_z P_U(z) \ll U^{1+M/N}, \qquad \text{absolute size } P_U(z) \ll U^{1+M/N}.$$

In addition

$$\log|P_U(x_0)| \ll -p(U)U^M \log U \ll -U^{1+M/N+M}.$$

Since $((U+1)/U)^{1+M/N}$ tends to 1, the hypotheses of Theorem 4.3 will be satisfied when $2\tau(M+N) < MN + M + N$. For then, if $C > 0$ is arbitrary, $\log|P_U(z)| \ll -U^{1+M/N+M} \leqslant -C(U^{1+M/N})^{2\tau}$ for large enough U. This is the desired contradiction, since x_0 is by assumption transcendental over K although Theorem 4.3 implies that it must be algebraic.

PROOF OF THEOREM B. Since this proof has much in common with the proof of Theorem A, we shall be somewhat briefer. Assume that the numbers $\alpha_i, \exp(\alpha_i\beta_j), i = 1, \cdots, M; j = 1, \cdots, N$, all lie in a field L of transcendence degree one over K, where L is as in (6.1). Let $I_L = Z[x_0, x_1, \cdots, x_m]$ and $A_i, C_{ij} \in I_L$ be denominators for multiplication by $\alpha_i, \exp(\alpha_i\beta_j)$, respectively, $1 \leqslant i \leqslant M$; $1 \leqslant j \leqslant N$ with respect to $(x_0, x_1, \cdots, x_m; 1, y, \cdots y^{l-1})$. Finally set $A = A_1 \cdots A_M$ and $C = \Pi C_{ij}$.

Let U be an integral parameter and set

$$\text{(6.17)} \qquad p(U) = [U^{(M+N)/(1+N)}/(\log U)^{N/(1+N)}].$$

Consider the exponential polynomial $f_U(z) = \Sigma A_\iota \exp(\alpha_\iota z)$ where $A_\iota \in I_L$ will be chosen later, $\iota = (i_1, \cdots, i_M)$, $\alpha_\iota = i_1\alpha_1 + \cdots + i_M\alpha_M$ and the sum is taken over all ι with $0 \leqslant i_1, \cdots, i_M < U$. We shall select the A_ι, not all zero, so that f_U has a zero of order at least $p(U)$ at every lattice point

$$j = j_1\beta_1 + \cdots + j_N\beta_N \tag{6.18}$$

with $0 \leqslant j_1, \cdots, j_N < [U^{(M-1)/(N+1)}(\log U)^{1/(N+1)}/(2^{(m+2)}l^2)^{1/N}]$.

We know that when j is given by (6.18), then each

$$A^{p(U)}C^{2[p(U)\log U]}\alpha_\iota^\sigma \exp(\alpha_\iota j), \tag{6.19}$$

$0 \leqslant \sigma < p(U)$, has matrix for multiplication with entries from I_L. Moreover (2.12) implies that (6.19) has

$$\text{size} \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)} \tag{6.20}$$

with respect to $(x_0, x_1, \cdots, x_m; 1, y, \cdots, y^{l-1})$. Thus when we evaluate

$$A^{p(U)}C^{2[p(U)\log U]}\alpha_\iota^p \exp(\alpha_\iota z),$$

$0 \leqslant p < p(U)$, at the lattice points in (6.18), we obtain elements of L whose multiplication matrices have entries from I_L with

$$\text{sizes} \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)}$$

Multiplying (6.19) by A_ι and setting the sums over ι equal to zero for each $0 \leqslant p < p(U)$ insures that f_U has zeros of order at least $p(U)$ at the lattice points of (6.18). However we cannot apply Lemma 5.2 directly. But since zero is the only element in L with multiplication matrix identically zero, setting each of the l^2 entries (they are of the form (5.5)) in the multiplication matrix for $f_U(j)$ equal to zero suffices. In this manner we get a system of homogeneous linear equations in the unknowns A_ι with coefficients in I_L having

$$\text{size} \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)}. \tag{6.21}$$

But the number of unknowns $= U^M$ and the

$$\begin{aligned}\text{number of equations} &\leqslant p(U) \cdot U^{(M-1)N/(N+1)}(\log U)^{N/N+1}/2^{m+2}\\ &\leqslant U^{(MN+M)/(N+1)}/2^{m+2}\\ &= U^M/2^{m+2}.\end{aligned}$$

Thus by Lemma 5.2, we know that there is nontrivial solution for the A_ι

in I_L such that each A_ι satisfies

$$\text{(6.22)} \qquad \text{size } A_\iota \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)}.$$

As before, we write for $U \geqslant U_1 \geqslant U_0$, using Lemma 5.4,

$$\text{(6.23)} \quad A_U = \frac{p_U!}{-4\pi^2} \int_\Gamma \frac{dz}{(z-\beta_U)^{p_U+1}} \int_{\Gamma^1} \prod \left(\frac{z-j}{\omega-j}\right)^{p(U)} \frac{f_U(\omega)}{\omega - z}\, d\omega,$$

where the product is over all lattice points of (6.18), Γ is $|z| = U^{(M-1/2)/(N+1)}$, and Γ^1 is $|\omega| = U^{M/(N+1)}$. Thus we find that

$$\begin{aligned}
\log |A_U| &\ll p_U \log p_U - p(U)(\log U) U^{(M-1)N/(N+1)}(\log U)^{1/(N+1)} \\
&\qquad + U \cdot U^{M/(N+1)} \\
&\ll - U^{(M+N)/(1+N)} \log U^{1/(1+N)} U^{(M-1)N/(N+1)} \log U^{1/(N+1)} \\
&\qquad + U^{(M+N+1)/(N+1)} \\
&\ll - U^M \log U,
\end{aligned}$$

since $(M+N+1) < MN + M$.

$$0 \neq B_U = A^{p(U)} C^{2[U^{(M+N)/(1+N)}]} A_U$$

satisfies

$$\log |B_U| \ll - U^M \log U, \quad \text{size } B_U \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)}.$$

Multiplying B_U together with all its conjugates over $\mathbf{Q}(x_0, x_1, \cdots, x_m)$ gives rise to $0 \neq P_U(x_0) \in I_L$,

$$\text{(6.24)} \qquad \log |P_U(x_0)| \ll - U^M \log U,$$

$$\text{(6.25)} \qquad \text{size } P_U(x_0) \ll U^{(M+N)/(1+N)}(\log U)^{1/(1+N)}.$$

If however $2\tau(M+N) \leqslant M(N+1)$, then $2\tau/(1+N) \leqslant M/(M+N) < 1$ and so

$$(U^{(M+N)/(1+N)}(\log U)^{1/(1+N)})^{2\tau}/U^M \log U \longrightarrow 0$$

as U tends to infinity. Thus (4.11) will be satisfied for large U, regardless of the constants implicit in (6.24) and (6.25). At the same time

$$((U+1)/U)^{(M+N)/(1+N)}(\log(U+1)/\log U)^{1/(1+N)} \longrightarrow 1,$$

so that (4.7) and (4.8) will be satisfied for any $a > 1$ if we start our sequence of $P_U(z)$'s with U large enough. Thus Theorem 4.3 forces x_0 to be algebraic over K. This contradiction establishes the theorem.

PROOF OF THEOREM C. Assume that the numbers $\exp(\alpha_i\beta_j)$, $1 \leqslant i \leqslant M$; $1 \leqslant j \leqslant N$, all lie in a field L of transcendence degree one over K, with L as in (6.1). Let $I_L = \mathbf{Z}[x_0, x_1, \cdots, x_m]$ and $C_{ij} \in I_L$ be a denominator for $\exp(\alpha_i\beta_j)$, $1 \leqslant i \leqslant M$; $1 \leqslant j \leqslant N$. Set $C = \Pi C_{ij}$.

Let U be an integral parameter and consider the exponential polynomial $f_U(z) = \Sigma A_\iota \exp(\alpha_\iota z)$, where $A_\iota \in I_L$ will be chosen, $\iota = (i_1, \cdots, i_M)$, $\alpha_\iota = i_1\alpha_1 + \cdots + i_M\alpha_M$ and the sum is taken over all ι with $0 \leqslant i_1, \cdots, i_M < U$. We shall select the A_ι, not all zero, so that $f_U(z)$ vanishes at every lattice point

$$j = j_1\beta_1 + \cdots + j_N\beta_N \tag{6.26}$$

with $0 \leqslant j_1, \cdots, j_N < [(U^M/2^{m+2}l^2)^{1/N}]$.

The elements of L

$$C^{[U^{1+M/N}]} f_U(j) \tag{6.27}$$

have matrices for multiplication which are homogeneous linear combinations with coefficients A_ι of products of matrices for C and the $C_{ij}\exp(\alpha_i\beta_j)$. Thus when we set all l^2 entries of the matrices for elements of the form (6.27) equal to zero, we have

$$l^2[(U^M/2^{m+2}l^2)^{1/N}]^N \leqslant [U^M/2^{m+2}] \quad \text{equations} \tag{6.28}$$

in the

$$U^M \quad \text{unknowns} \tag{6.29}$$

A_ι. Moreover by (2.12), the coefficients have size $\ll U^{1+M/N}$. Lemma 5.2 assures a nontrivial solution for the $A_\iota \in I_L$ satisfying

$$\text{size } A_\iota \ll U^{1+M/N}. \tag{6.30}$$

Thus $f_U(z)$ is not identically zero since $\alpha_1, \cdots, \alpha_M$ are linearly independent over $\mathbf{Q}$. Since $\beta_1, \cdots, \beta_N$ are linearly independent over $\mathbf{Q}$, Lemma 5.4 shows that when $U \geqslant U_0$, $f_U(z)$ does not vanish at some lattice point

$$\beta_U = j_1\beta_1 + \cdots + j_N\beta_N \tag{6.31}$$

where $0 \leqslant j_1, \cdots, j_N < 2[U^{M/N}]$. We estimate the function

$$g_U(z) = \left(\prod(\beta_U - j)/\prod(z - j)\right) f_U(z),$$

on a circle of radius $R = U^{3M/2N}$ where the products run over all j of (6.26). In light of (6.30), when U is large,

$$\log |g_U(z)|_{|z|=R} \ll - \tfrac{1}{4}(M/N)(\log U)U^M + U^{1+M/N} \ll - U^M \log U.$$

Thus by the maximum modulus principle,

$$\log |f_U(\beta_U)| = \log |g_U(\beta_U)| \leqslant \log |g_u(z)|_{|z|=R} \ll - U^M \log U. \tag{6.32}$$

However size $f_U(\beta_U) \ll U^{1+M/N}$.

Now $0 \neq B_U = C^{2U[U^{M/N}]} f_U(\beta_U)$, but

$$\log |B_U| \ll - U^M \log U, \quad \text{size } B_U \ll U^{1+M/N}.$$

Multiplying B_U together with all its conjugates over $\mathbf{Q}(x_0, x_1, \cdots, x_m)$ gives rise to $0 \neq P_U(x_0) \in I_L$,

$$\log |P_U(x_0)| \ll - U^M \log U, \tag{6.33}$$

$$\text{size } P_U(x_0) \ll U^{1+M/N}. \tag{6.34}$$

If, however, $2\tau(M + N) \leqslant MN$, then $U^{2\tau(1+M/N)}/U^M \log U \longrightarrow 0$ as U tends to infinity and (4.11) will be satisfied for large U regardless of the constants implicit in (6.33) and (6.34). At the same time $(U + 1)^M \log(U + 1)/U^M \log U \longrightarrow 1$, so that (4.7) and (4.8) will be satisfied for any $a > 1$ if we start our sequence of $P_U(z)$'s with U large enough. Theorem 4.3 then furnishes the desired contradiction.

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